Explicit Crack Problem Solutions of Unidirectional Composites: Elastic Stress Concentrations

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This paper examines the stress concentration problems in unidirectional continuous fiber composites due to transverse multifilament failure. The loading conditions are considered: uniform load at infinity and concentrated force dipole on the crack plane. The boundary value problems are formulated based upon a two-dimensional elastic shear-lag model. The present theory is unique in that the general solution for the multifilament failure problem is obtained explicitly using the Legendre polynomials and that the stress concentration factors in all fibers on the crack plane are given in closed forms. The present analysis also provides rigorous proofs of Hedgepeth's inspection on the tensile stress concentration factor and Fichter's inspection on the shear stress concentration factor at the tip of a notch.

Nomenclature

b = total number of broken fibers
d = effective fiber spacing
EA = fiber extensional stiffness
F_n = nondimensionalized axial force
G = effective shear modulus
G_{nm} = $-L_{nm}$
h = thickness of composite per one fiber
K_b^s = stress concentration factor in the $(b+s)$-th fiber
$K_b^{s,t}$ = stress concentration factor at the s-th fiber from the tip of a crack containing b broken fibers, due to a unit applied force at the $(b\text{-}t)$-th fiber
L_{nm} = axial stress in the n-th fiber due to a unit displacement at the m-th fiber
L^*_{nm} = displacement of the n-th fiber on the crack plane
n = designation of fiber number
P = unit force dipole
$P_n(y)$ = reversible force in the n-th fiber
s = designation of fiber number at the tip of the crack
$S_{\max}$ = maximum shear stress concentration factor
$u_n(y)$ = fiber displacement
U_n = nondimensionalized displacement
y = coordinate in the fiber direction
ξ = nondimensionalized coordinate
$\tilde{\tau}_n(y)$ = matrix shear force per unit length

Introduction

FIBER reinforced composite materials have gained increasing technological importance due to the versatility in properties and high performance.[1,2] The strength of composites is often affected by fiber breakages due to cracks and cut-outs. It is thus essential to understand the local stress concentrations induced by fiber fractures.

The problem of stress concentration in composites has been treated by the shear-lag method,[3-11] elasticity theory[12,13] and numerical methods.[14,15] Among these approaches, the shear-lag approach, which is based upon simplified assumptions, often provides good physical insights of rather complex problems. The shear-lag method was first adopted by Hedgepeth[3] to treat multifilament failure problems of laminated composites. The technique also has been extended to include the effects of plasticity of the matrix,[4,9] and the condition of interfacial debonding.[5]

In this paper, we examine the two-dimensional multifilament failure problem of unidirectional fiber composites, focusing specifically on the stress concentration factors of fibers adjacent to the cracks. The physical problems are analyzed by the two-dimensional shear-lag method under two loading conditions:

a) uniform tensile force applied to all fibers at infinity (Fig. 1)

b) concentrated force dipole applied at a particular fiber, $n = b - t$, on the crack plane (Fig. 2).

The present analyses are unique in that the general solution of the governing equations of the elastic field has been obtained in explicit forms in terms of the Legendre polynomials for the loading condition (A). Based upon this solution, closed form expressions of stress concentration factors in all fibers

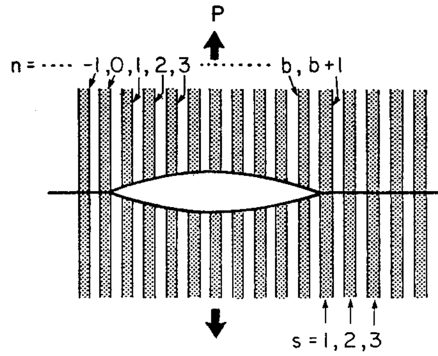

Fig. 1 Model of a multifilament crack in a unidirectional composite under uniform force at infinity.

Received Sept. 13, 1988; revision received May 17, 1989.

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have been derived. Furthermore, we also present rigorous proofs of both Hedgepeth's inspection[4] on the general form of the tensile stress concentration factor at the tip of a crack and Fichter's inspection[8] on the general form of the shear stress concentration factor for the loading condition (A).

Since there exists a reciprocal relation between the influence function matrices for the loading condition (A) and (B), the solution for the condition (B) can be readily derived from the solution for the condition (A).

The stress distribution obtained by the present theory gives more detailed information on the state of deformation around the crack tip than that obtained by treating the composite as a homogeneous medium. It also provides a sound basis for studying the statistical aspects of composite strength and failure.

General Formulation

The analysis considers a unidirectional continuous fiber composite containing a slit notch in the transverse direction as shown in Fig. 1. The fiber direction is taken along the y axis. The broken fibers are denoted as $n = 1,2,3, \ldots ,b$, starting from the left tip of the notch with b being the total number of fibers in the notch.

Under the assumption of shear-lag analysis,[3] the matrix material transfers only shear force, $\bar{\tau}_n(y)$, per unit length between two adjacent fibers. Thus $\bar{\tau}_n(y)$ is related to the difference of displacements $u_n(y)$ in the fiber direction as

$$\bar{\tau}_n(y) = (Gh/d)[u_{n+1}(y) - u_n(y)] \tag{1}$$

where G is the effective shear modulus of the matrix, h is the thickness of the composite per one fiber, and d is the effective fiber spacing. The tensile force $Pn(y)$ in the n-th fiber is related to the displacement by

$$Pn(\mathrm{y}) = EA\frac{\mathrm{d}u_n(\mathrm{y})}{\mathrm{d}y} \tag{2}$$

where EA is the extensional stiffness of the fibers.

Following Hedgepeth,[4] the nondimensionalized axial force, displacement, and coordinate are given, respectively, by:

$$F_n(\xi) = P_n(y)/P, \qquad U_n(\xi) = u_n(y)(EAGh/dP^2)^{1/2} \tag{3}$$

$$\xi = (Gh/EAd)^{1/2} \cdot y$$

Then, the equilibrium equations can be written as

$$\frac{\mathrm{d}^2U_n(\xi)}{\mathrm{d}\xi^2} = 2U_n(\xi) - U_{n+1}(\xi) - U_{n-1}(\xi) \tag{4}$$

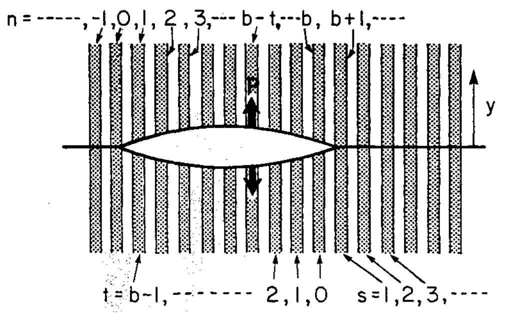

Fig. 2 Model of a multifilament crack in a unidirectional composite under concentrated force dipole in the $(b-t)$-th fiber on the crack plane.

The boundary conditions are as follows:

$$F_n(0) = 0, \qquad (1 \le n \le b) \tag{5a}$$

$$U_n(0) = 0, \qquad (n \le 0,\ n \ge b+1) \tag{5b}$$

$$F_n(\pm\infty) = 1, \qquad (\text{all } n) \tag{5c}$$

for loading condition (A), and

$$F_n(0) = 0, \quad (n = 1,2,\cdots,\ b-t-1, b-t+1,\cdots,b) \tag{6a}$$

$$U_n(0) = 0, \qquad (n \le 0,\ n \ge b+1) \tag{6b}$$

$$F_n(\pm\infty) = 0. \qquad (\text{all } n) \tag{6c}$$

$$F_{b-t}(0) = -1 \tag{6d}$$

for loading condition (B).

Loading Condition (A)

Analysis

Hedgepeth[3] solved the problem for the loading condition (A) using Fourier transformation in the following manner.

$$U_n(\xi) - \xi = \frac{1}{2\pi}\int_0^{2\pi} \bar{V}(\theta,\xi)e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta, \qquad (n = 0,\ \pm 1,\ \pm 2\ldots) \tag{7}$$

$$\bar{V}(\theta,\xi) = e^{-\gamma|\xi|}\sum_{n=1}^{b} U_n(0)e^{in\theta} \tag{8}$$

where

$$\gamma = 2\left|\sin\frac{\theta}{2}\right| \tag{9}$$

The unknown constant $U_n(0)$ is obtained by solving the following linear equation.

$$[\tilde{\tilde{G}}]\begin{bmatrix} U_1(0) \\ U_2(0) \\ \cdot \\ \cdot \\ U_b(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{10a}$$

where the components of the matrix $[\tilde{\tilde{G}}]$ are

$$G_{nm} = \frac{4}{\pi}\frac{1}{[1-4(m-n)^2]} \tag{10b}$$

In Hedgepeth's paper[4] the influence function L_{nm} is defined as the axial stress in the n-th fiber when a unit displacement takes place at the m-th fiber, and $L_{nm} = -G_{nm}$. The inverse matrix of $[\tilde{\tilde{G}}]$ in Eq. (10) for an arbitrary number of broken fibers is difficult to obtain; however, the general form of $U_n(0)$ is obtained explicitly using Legendre polynomials

$$U_n(0) = \pi\{2(b-n)+1\}!\{2n-1\}!/2^{2b}\{(b-n)!(n-1)!\}^2 \tag{11}$$

In the following it is shown that $U_n(0)$ in Eq. (11) is the solution for the linear equation (10a).

Proof

The m-th component of the left side of Eq. (10a) is given using Eq. (11)

$$I_m = \sum_{k=1}^{b} G_{mk}\cdot U_k(0)$$

$$= \sum_{k=1}^{b}\left\{\frac{1}{2\pi}\int_0^{2\pi}\gamma e^{-i(m-k)\theta}\,\mathrm{d}\theta\right\}\cdot\frac{\pi\{2(b-k)+1\}!\{2k-1\}!}{2^{2b}\{(b-k)!(k-1)!\}^2}$$

$$= \frac{1}{2}\int_0^{\pi}\mathrm{d}\phi\ \{\sin\phi\ \cos(b+1-2m)\phi\}\ f_{b-1}(\cos\phi) \tag{12}$$

where

$$f_{b-1}(\cos\phi) \equiv \sum_{k=0}^{b-1} \frac{\{2(b-k-1)+1\}!\{2k+1\}!}{2^{2(b-1)}\{(b-k-1)!k!\}^2} e^{i(2k-b+1)\phi} \quad (13)$$

Equation (12) is further transformed into Eq. (14).

$$I_m = \frac{1}{2}\int_0^{\pi} d\phi\ \{\sin\phi \cos(b+1-2m)\phi\} b^2 P_{b-1}(\cos\phi)$$

$$+\frac{1}{2}\int_0^{\pi} d\phi \left\{\frac{d^2}{d\phi^2}(\sin\phi \cos(b+1-2m)\phi)\right\} P_{b-1}(\cos\phi)$$

$$-\frac{1}{2}\left[\left\{\frac{d}{d\phi}\{\sin\phi \cos(b+1-2m)\phi\}\right\} P_{b-1}(\cos\phi)\right]_0^{\pi}$$

$$= 1 + \frac{(b-\ell-1)(b+\ell+1)}{4}\int_0^{\pi} \sin(\ell+1)\phi\ P_{b-1}(\cos\phi)\, d\phi$$

$$+\frac{(\ell-1-b)(\ell-1+b)}{4}\int_0^{\pi} \sin(\ell-1)\phi\ P_{b-1}(\cos\phi)\, d\phi \quad (14)$$

where $\ell = b+1-2m$. In the derivations of Eq. (14), the following equations are used for relating $f_{b-1}(\cos\phi)$ and the Legendre polynomials $P_{b-1}(\cos\phi)$.

$$\left(\frac{d}{id\phi}+b\right)P_{b-1}(\cos\phi)$$

$$= \sum_{r=0}^{b-1} \frac{1}{2^{2(b-1)}} \frac{\{2(b-r-1)\}!\{2r\}!}{\{(b-r-1)!r!\}^2} \cdot (2r+1)\, e^{i(2r-b+1)\phi} \quad (15)$$

$$\left(-\frac{d}{id\phi}+b\right)\left(\frac{d}{id\phi}+b\right)P_{b-1}(\cos\phi)$$

$$= \sum_{r=0}^{b-1} \frac{1}{2^{2(b-1)}} \frac{\{2(b-r-1)+1\}!\{2r+1\}!}{\{(b-r-1)!r!\}^2} e^{i(2r-b+1)\phi}$$

$$= \frac{4}{\pi}\cdot \bar{V}(\theta,0)\cdot e^{-i(1+b)\theta} \quad (16)$$

where $\phi = \theta/2$ and

$$P_{b-1}(\cos\phi) = \sum_{r=0}^{b-1} \frac{1}{2^{2(b-1)}} \frac{\{2(b-r-1)\}!\{2r\}!}{\{(b-r-1)!r!\}^2} e^{i\{(2r-b+1)\}\phi} \quad (17)$$

Furthermore, by using Eq. (18), the last two integrals in Eq. (14) are shown to be zero for $m = 1, 2, \ldots, b$.

$$\int_0^{\pi} \sin n\phi\ P_m(\cos\phi)\, d\phi = \frac{2(n-m+1)(n-m+3)\cdots(n+m-1)}{(n-m)(n-m+2)\cdots(n+m)} \quad (18)$$

for $n > m$ and $n+m$ is odd

$= 0$

for $h \leq m$ or $n+m$ is even

Thus, the expression $U_n(0)$ is Eq. (11) is proven to be the solution of Eq. (10a).

Using Eq. (11), $U_n(\xi)$ is obtained from Eqs. (7) and (8)

$$U_n(\xi) = \xi + \frac{1}{2}\int_0^{2\pi} d\theta e^{-in\theta-\gamma\xi} \times \left\{\sum_{m=1}^{b} \frac{[2(b-m)+1]![2m-1]!}{2^{2b}[(b-m)!(m-1)!]^2} e^{im\theta}\right\} \quad (19a)$$

and

$$F_n(\xi) = \frac{dU_n(\xi)}{d\xi} \quad (19b)$$

Furthermore, the above solution can be expressed in terms of Legendre polynomials, $P_n(\cos\phi)$, as follows ($\xi \geq 0$)

$$U_n(\xi) = \xi + \frac{1}{4}\int_0^{\pi} d\phi\ e^{(b+1-2n)i\phi-\gamma\xi}\left\{b^2+\frac{d^2}{d\phi^2}\right\}P_{b-1}(\cos\phi)\, d\phi \quad (20)$$

$$F_n(\xi) = 1 - \frac{1}{4}\int_0^{\pi} d\phi\gamma\ e^{(b+1-2n)i\phi-\gamma\xi}\left\{b^2+\frac{d^2}{d\phi^2}\right\}P_{b-1}(\cos\phi)\, d\phi \quad (21)$$

In deriving Eq. (20), Eqs. (15) and (16) have been used.[16]

Stress Concentration on the Crack Plane

The stress concentration factors of fibers ahead of the crack tip on the crack plane are derived in the following. The stress concentration factor in the $(b+s)$-th fiber is denoted as K_b^s. From the general solution of Eq. (21) we obtain

$$K_b^s \equiv F_{b+s}(0)$$

$$= 1 - \frac{1}{2}\int_0^{\pi} \sin\phi \cdot e^{-i(b+2s-1)\phi}\cdot\left\{b^2+\frac{d^2}{d\phi^2}\right\}P_{b-1}(\cos\phi)\, d\phi \quad (22)$$

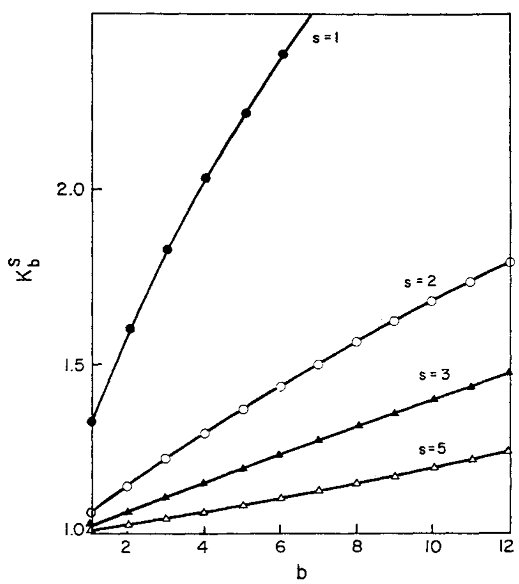

Fig. 3 Stress concentration factor K_b^s in the $(b+s)$-th fiber. b denotes the number of broken fibers. $s=1$ corresponds to the special case of Hedgepeth.

Carrying out the above integration by parts, the closed-form expression for the stress concentration factor is obtained

$$K_b^s = (b+2s-1)\frac{2s\cdot(2s+2)\cdot(2s+4)\cdots(2s+2b-2)}{(2s-1)\cdot(2s+1)\cdot(2s+3)\cdots(2s+2b-3)\cdot(2s+2b-1)} \tag{23}$$

Here, we have used the relation of Eq. (18).

As a special case of Eq. (23), the stress concentration factor in the first intact fiber (s = 1) adjacent to b broken fibers is

$$K_b^1 = \frac{4\cdot 6\cdot 8\cdots(2b+2)}{3\cdot 5\cdot 7\cdots(2b+1)} \tag{24}$$

Hedgepeth[3] deduced Eq. (24) by inspecting the numerical results of the case $b = 1,2, \ldots ,6$. Here, his inspection on the general form of the stress concentration factor has been rigorously proven. Fig. 3 depicts the numerical results for K_b^s.

Furthermore, the maximum shear stress takes place in the matrix at the tip of the crack. Thus, the dimensionless displacement at the crack tip $U_b(0)$ is termed the maximum shear stress concentration factor, $S_{\max}$. By using Eq. (11), we obtain

$$S_{\max} = \frac{\pi(2b-1)!}{2^{2b}\{(b-1)!\}^2} \tag{25}$$

Fichter[8] also deduced the above result by calculating the cases of $b = 1, 2, \ldots ,6$, and our solution also gives the rigorous proof of his inspection.

Fiber Stress Away from the Crack Plane

The axial stress in fibers away from the crack plane can also be obtained from Eq. (21) and expressed as a series expansion in terms of the coordinate ξ

$$F_{b+s}(\xi) = K_b^s + \sum_{k=1}^{\infty} C_k |\xi|^k \tag{26}$$

where

$$C_k = 2^{2p}\cdot(2p+1)\cdot\left\{\sum_{\substack{m=1\\ p\geqq|m-n|-1}}^{b} \frac{(-1)^{m-n+1}\cdot\{2(b-m)+1\}!!\cdot\{2m-1\}!!}{\{2p+1-2|m-n|\}!!\cdot\{2p+1+2|m-n|\}!!\cdot\{2b-2m\}!!\cdot\{2m-2\}!!}\right.$$

$$\left. + \sum_{\substack{m=1\\ p<|m-n|-1}}^{b} \frac{(-1)^p\cdot\{2|m-n|-2p-3\}!!\cdot\{2(b-m)+1\}!!\cdot\{2m-1\}!!}{\{2p+1+2|m-n|\}!!\cdot\{2b-2m\}!!\cdot\{2m-2\}!!}\right\} \quad (\text{for } k\text{: even} = 2p) \tag{27a}$$

and

$$C_k = \frac{\pi}{2}\sum_{\substack{m=1\\ p\geqq|m-n|}}^{b} (-1)^{m-n}\cdot\frac{p\cdot\{2(b-m)+1\}!!\cdot\{2m-1\}!!}{\{p+|m-n|\}!\cdot\{p-|m-n|\}!\cdot\{2b-2m\}!!\cdot\{2m-2\}!!} \quad (\text{for } k\text{: odd} = 2p-1) \tag{27b}$$

where $n!!$ denotes the double factorial.

Loading Condition (B)

Analysis

As shown in Fig. 2, we assume a unit force dipole is applied on the a-th fiber ($a = b - t$). The more general cases with multiple dipoles are obtained by the linear combination of this simple solution.

We denote the co-influence coefficient as L_{nm}^* which represents the displacement of the n-th fiber when the unit force dipole is applied on the m-th fiber at the crack plane. It is obvious that the matrix $[\tilde{\tilde{L}}^*]$ is the inverse of the matrix $[\tilde{\tilde{L}}]$

$$[\tilde{\tilde{L}}]\cdot[\tilde{\tilde{L}}^*] = [\tilde{\tilde{L}}^*]\cdot[\tilde{\tilde{L}}] = [\tilde{\tilde{E}}] \tag{28}$$

where the matrix $[\tilde{\tilde{L}}^*]$ is composed of the components L_{nm}^* and the matrix $[\tilde{\tilde{L}}]$ is composed of the component L_{nm}. The matrix $[\tilde{\tilde{E}}]$ represents the $(b\times b)$ unit matrix. Thus the stress in the c-th fiber ($c = b+s$) on the crack plane is obtained as follows.

$$K_{a,b}^c = -\sum_{k=1}^{b} L_{ck}\cdot L_{ka}^* = \frac{(-1)}{\det[\tilde{\tilde{L}}]}\det\begin{bmatrix} L_{1,1} & L_{1,2} & \cdot & \cdot & \cdot & L_{1,b}\\ L_{2,1} & L_{2,2} & \cdot & \cdot & \cdot & L_{2,b}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ L_{a-2,1} & L_{a-2,2} & \cdot & \cdot & \cdot & L_{a-2,b}\\ L_{a-1,1} & L_{a-1,2} & \cdot & \cdot & \cdot & L_{a-1,b}\\ L_{c,1} & L_{c,2} & \cdot & \cdot & \cdot & L_{c,b}\\ L_{a+1,1} & L_{a+1,2} & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ L_{b,1} & L_{b,2} & \cdot & \cdot & \cdot & L_{b,b} \end{bmatrix} \tag{29}$$

The determinant in Eq. (29) can be evaluated as follows.

Consider the following relation between the influence function L_{nm}, the fiber displacements on the crack plane, and the stress concentration factors under the loading conditon (A)

$$\sum_{k=1}^{b} L_{nk}U_k^b = \begin{cases} -1 & (n\leq b)\\ K_b^{n-b}-1 & (n>b)\end{cases} \tag{30}$$

U_k^b represents the displacement in the k-th fiber on the crack plane under the loading condition (A), when b fibers are broken.

For the special case of $a = b = c$, we can evaluate the determinant of the matrix $[\tilde{\tilde{L}}]$ in terms of U_b^b and K_b^1.

$$\det[\tilde{\tilde{L}}] = (-1)^b\frac{\prod_{i=1}^{b-1}K_i^1}{\prod_{i=1}^{b}U_i^i} = (-1)^b\frac{\prod_{i=1}^{b-1}\left\{\frac{1}{2}\cdot\frac{(2i+2)!!}{(2i+1)!!}\right\}}{\prod_{i=1}^{b}\left\{\frac{\pi}{4}\cdot\frac{(2i-1)!!}{(2i-2)!!}\right\}} \tag{31}$$

Using Eq. (31), the stress concentration factor $K_{a,b}^c$ is reduced to the following more simplified form.

$$K_{a,b}^{c}=\frac{1}{K_{a-1}^{1}\cdot K_{a}^{1}\cdot{}_{a+1}^{1}\cdots K_{b-1}^{1}}\quad\det\left[\begin{array}{ccccc} K_{b-1}^{c-b+1}-k_{b}^{c-b} & K_{b-2}^{c-b+2}-k_{b}^{c-b} & \cdot & \cdot & K_{a-1}^{c-a+1}-k_{b}^{c-b} \\ 0 & 0 & \cdot & K_{a}^{1} & K_{a-1}^{2} \\ 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & K_{b-2}^{1} & \cdot & \cdot & \cdot \\ K_{b-1}^{1} & K_{b-2}^{2} & \cdot & \cdot & k_{a-1}^{b-a+1} \end{array}\right]$$

$$(b-a+1)$$

$$=\det\left[\begin{array}{cccccc} \frac{(K_{b-1}^{s+1}-k_{b}^{s})}{K_{b-t-1}^{1}} & \frac{(K_{b-2}^{s+2}-K_{b}^{s})}{K_{b-t-1}^{1}} & \cdot & \cdot & \cdot & \frac{(K_{b-t-1}^{s+t+1}-K_{b}^{s})}{K_{b-t-1}^{1}} \\ 0 & 0 & 0 & 0 & 1 & \frac{b-t+2}{b-t+1}\,\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & \frac{b}{b-1}\cdot\frac{1!!}{2!!} & \cdot & \cdot & \frac{b+t-2}{b-1}\cdot\frac{(2t-3)!!}{(2t-2)!!} \\ 1 & \frac{b+1}{b}\cdot\frac{1!!}{2!!} & \frac{b+2}{b}\cdot\frac{3!!}{4!!} & \cdot & \cdot & \frac{b+t}{b}\cdot\frac{(2t-1)!!}{(2t)!!} \end{array}\right] \tag{32}$$

Finally, the above expression is further reduced to the following closed form

$$K_{b}^{s,t}\equiv K_{a,b}^{c}=\frac{1}{2}\,\frac{(2t+1)!!\cdot(2a-1)!!\cdot(2s-3)!!\cdot(2c-2)!!}{(2t)!!(2a-2)!!\cdot(2s-2)!!\cdot(2c-1)!!}\cdot\frac{1}{(s+t)} \tag{33}$$

where $t=b-a$, $s=c-b$. The proof of Eq. (33) is given in the following section entitled "Proof of Eq. 33".

The highest fiber stress concentration takes place at the edge of the crack ($s=1$),

$$K_{b}^{1,t}=\frac{(2t+1)!!\cdot(2a-1)!!\cdot(2b)!!}{(2t+2)!!\cdot(2a-2)!!\cdot(2b+1)!!} \tag{34}$$

Fig. 4 depicts the numerical results for $K_{b}^{1,t}$. For a semi-infinite crack the stress concentration factor at the s-th fiber

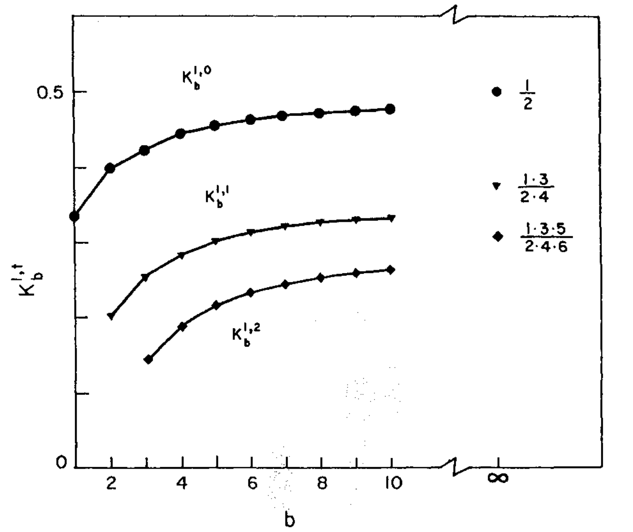

Fig. 4 Stress concentration factor $K_{b}^{1,t}$ in the $(b+1)$-th fiber when the unit load is applied at the $(b\text{-}t)$-th fiber. b denotes the number of broken fibers.

from the crack tip due to the unit applied force dipole at the t-th fiber has the following value

$$\lim_{b\to\infty}K_{b}^{s,t}=\frac{1}{2(s+t)}\cdot\frac{(2t+1)!!}{(2t)!!}\cdot\frac{2s-3)!!}{(2s-2)!!} \tag{35}$$

The general solution for the stress in the fibers can be obtained from Eq. (33)

$$F_{n}(\xi)=\frac{1}{2\pi}\int_{0}^{2\pi}d\theta\; e^{-in\theta-\gamma\xi}\{-e^{ia\theta}+\sum_{c\neq a}\dot{K}_{a,b}^{c}\,e^{ic\theta}\}\;(\xi\geqq 0) \tag{36a}$$

where

$$\dot{K}_{a,b}^{c}=\begin{cases}K_{a,b}^{c} & \text{for } c\geqq b+1\\ K_{b+1-a,b}^{b+1-c} & \text{for } c\leqq 0\\ 0 & \text{for } 1\leqq c\leqq b\end{cases} \tag{36b}$$

Proof of Eq. (33)

The determinant of the matrix [Eq. (32)] can be calculated by subtracting the nonzero components as follows:

$$K_{b}^{s,t}=\frac{1}{K_{b-t-1}^{1}}\det\left[\begin{array}{cccccccc} d_{1}' & d_{2}' & \cdot\,\cdot & 0 & d_{t}' & d_{t+1}' & & \\ 0 & 0 & \cdot\,\cdot & 0 & 1 & 0 & & \\ 0 & 0 & \cdot & 0 & 1 & 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \end{array}\right]$$

$$=\frac{|d_{t+1}'|}{K_{b-t-1}^{1}}. \tag{37}$$

In the following, using the inductive method, we prove that d_{k}' ($k=1,2,\ldots,t+1$) is expressed as in Eq. (38) for fixed values of b, s, and t.

$$d_k' = -\frac{1}{2}\cdot\frac{3\cdot5\cdots(2k-1)}{2\cdot4\cdots(2k-2)}\cdot\frac{b-k+1}{s+k-1}\cdot\frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)} \tag{38}$$

Proof. For k = 1, by using Eq. (23),

$$d_1' = k_{b-1}^{s+1} - K_b^s = (b+2s)\frac{(2s+2)\cdots(2s+2b-2)}{(2s+1)\cdots(2s+2b-1)}$$

$$-(b+2s-1)\frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)}$$

$$= -\frac{1}{2}\frac{b}{s}\frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)}. \tag{39}$$

Thus, for $k = 1$ Eq. (38) is proven to be true.

Then, assume d_k' is expressed as in Eq. (38) for $k = 1,2,\ldots,n$. For $k = n+1$, we calculate J as follows

$$J = d_{n+1}' + \frac{1}{2}\cdot\frac{3\cdot5\cdots(2n+1)}{2\cdot4\cdots(2n)}\cdot\frac{b-n}{s+n}\cdot\frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)} \tag{40}$$

d_{n+1}' is related to d_i' ($i = 1,2,\ldots,n$) by the following Eq. (41) [see Eq. (32)]

$$d_{n+1}' = K_{b-n-1}^{s+n+1} - K_b^s - \sum_{k=1}^{n}\frac{b+n-2k+2}{b-k+1}\cdot\frac{(2n-2k+1)!!}{(2n-2k+2)!!}d_k' \tag{41}$$

Thus,

$$J = K_{b-n-1}^{s+n+1} - K_b^s - \sum_{k=1}^{n}\frac{b+n-2k+2}{b-k+1}\cdot\frac{1\cdots(2n-2k+1)}{2\cdots(2n-2k+2)}d_k' + \frac{1}{2}\cdot\frac{3\cdot5\cdots(2n+1)}{2\cdot4\cdots(2n)}\frac{b-n}{s+n}\cdot\frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)}$$

$$= \frac{(2s)\cdots(2s+2b-2)}{(2s-1)\cdots(2s+2b-1)}\left\{(b+2s+n)\frac{(2s-1)\cdot(2s+1)\cdots(2s+2n-1)}{(2s)\cdot(2s+2)\cdots(2s+2n)} - (b+2s-1)\right.$$

$$\left. + (b+n+2s)\cdot\sum_{k=0}^{n}\frac{3\cdot5\cdots(2k+1)}{2\cdot4\cdots(2k)}\cdot\frac{1}{2(s+k)}\cdot\frac{1\cdots(2n-2k-1)}{2\cdots(2n-2k)} - \sum_{k=0}^{n}\frac{1\cdot3\cdots(2k-1)}{2\cdot4\cdots(2k)}\cdot\frac{1\cdots(2n-2k-1)}{2\cdots(2n-2k)}\cdot(2k+1)\right\} \tag{42}$$

The third and fourth terms in Eq. (42) are calculated by using the Legendre polynomial integrals,[17]

$$\sum_{k=0}^{n}\frac{3\cdot5\cdots(2k+1)}{2\cdot4\cdots(2k)}\cdot\frac{1}{2(s+k)}\cdot\frac{1\cdots(2n-2k-1)}{2\cdots(2n-2k)} = \int_{\infty}^{0}\frac{d}{dz}\{P_n(\cosh z)\cdot e^{-(n+1)z}\}\cdot e^{-(2s-1)z}\,dz$$

$$= P_n(1) - (2s-1)\int_0^{\infty}P_n(\cosh z)\cdot e^{-(n+2s)z}\,dz = 1 - \frac{(2s-1)\cdot(2s+1)\cdot(2s+3)\cdots(2s+2n-1)}{(2s)\cdot(2s+2)\cdot(2s+4)\cdots(2s+2n)} \tag{43}$$

and

$$\sum_{k=0}^{n}\frac{1\cdot3\cdots(2k-1)}{2\cdot4\cdots(2k)}\cdot\frac{1\cdots(2n-2k-1)}{2\cdots2(n-k)}\cdot(2k+1) = -\left[\frac{d}{dz}\{P_n(\cosh z)e^{-(n+1)z}\}\right]_{z=0} = n+1 \tag{44}$$

In calculating the above equation, the following relations are used

$$P_n(\cosh z) = \sum_{k=0}^{n}\frac{2(n-k)-1\cdots1}{2(n-k)\cdots2}\cdot\frac{(2k-1)\cdots1}{(2k)\cdots2}\cdot e^{-(2k-n)z} \tag{45}$$

$$\int_0^{\infty}P_n(\cosh z)e^{-az}\,dz = \frac{(a-n+1)(a-n+3)\cdots(a+n-1)}{(a-n)(a-n+2)\cdots(a+n)}\quad (Re\ a > n) \tag{46}$$

By putting Eqs. (43) and (44) into (42), J is proven to be zero. Hence, for $k = n+1$, Eq. (38) also is proven to be true. Thus it is proven that d_k' generally can be expressed by Eq. (38). Finally, by Eq. (37), $K_b^{s,t}$ is expressed in the simple formula of Eq. (33).

Conclusion

A study of the two dimensional transverse crack problem in unidirectional fiber composites has been performed. Analysis of the problem is based on the shear lag approach of elastic materials.

Explicit solutions of the elastic field are obtained for an arbitrary size of the crack under two loading conditions: uniform tensile force at infinity and force dipole on the crack surfaces.

Stress concentration factors for all unbroken fibers on the crack plane also have been obtained in closed forms. These results have been compared with the special cases of Hedgepeth and Fichter.

Acknowledgment

We wish to thank Dr. Kunio Nishioka of Narumi China Corporation, for the support and encouragement given to one of the authors, Fuminori Hikami. Tsu-Wei Chou thanks the office of Naval Research for partial support; Dr. L. H. Peebles, Jr. is the program manager.

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